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Author(s)	Tanaka, Keiichi
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Value Function of Real Options with Regime Switching

Keiichi Tanaka *
Tokyo Metropolitan University

1 Introduction

We consider irrevesible investment problems with regime switching feature under a monopoly setting. Several parameters describing the economic environment varies according to a regime switching with general number of states. We present the derivation of the value function via solving a system of simultaneous ordinary differential equations with knowledge of linear algebra. It enables us to investigate a comparative analysis of the investment problem. The contribution of this paper is a natural extension of Guo and Zhang (2004) to cases of general number of regime states in the context of real options.

2 Setup

In this paper a matrix is represented in bold. $\mathbf{O}_n$ denotes the zero matrix of order n and $\mathbf{I}_n$ denotes the identity matrix of order n.

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on infinite time horizon. Let $J = \{J(t)\}$ be a continuous-time Markov chain on a state space $E = \{1, 2, \cdots, S\}$. $J(t)$ is interpreted as a regime or a state of the economy at time t. The intensity matrix of the regime is given by $\mathbf{Q}$

$$\mathbf{Q} = (q_{ij})_{i,j \in E}, \quad q_{ii} = -\sum_{j \in E \setminus \{i\}} q_{ij}.$$

The process $X = \{X_t\}$ satisfies

$$dX_t = \mu_{J(t)} X_t dt + \sigma_{J(t)} X_t dW_t, \quad X_0 = x,$$

where $W = \{W_t\}$ is a standard Brownian motion, μ_j and σ_j are constants for each $j \in E$. Denote the filtration genereated by (W, J) as $\{\mathcal{F}_t\}$ with $\mathcal{F}_t = \sigma(W_s, J(s), 0 \leq s \leq t)$.

The firm has a chance to start a project to make a product as a monoply of the product whose revenue depends on the state variables $(X_t, J(t))$ of the economy. We assumes that the firm obtains the instant revenue of $D_i X_t$ at time t from the project when the regime state is i. D_i $(i \in E)$ is a positive constant.

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The firm has a technology to enter into the project by paying the cost K_i when the regime state is i. When the current regime state is i, the value function V_i is defined by

$$V_i(x) = \max_{\tau} E\left[\int_{\tau}^{\infty} e^{-ru} D_{J(u)} X_u du - e^{-r\tau} K_{J(\tau)} \mid X_0 = x, J(0) = i\right].$$

Let us denote a vector and a matrix

$$\mathbf{D} = \begin{pmatrix} D_1 & \cdots & D_S \end{pmatrix}^{\top}, \quad \mathbf{M} = \operatorname{diag}[\mu_1, \cdots, \mu_S].$$

For simple notation it is convenient to introduce $\mathbf{H}_n$ a "truncationg" operator on $S \times S$ square matrix $\mathbf{A}$

$$\mathbf{H}_n((a_{ij})_{1 \le i,j \le S}) = (a_{ij})_{1 \le i,j \le n}.$$

We assume the following properties;

Assumption 1. 1. $\mathbf{Q}$ is irreducible.

2. The matrix $r\mathbf{I}_S - \mathbf{M}_S - \mathbf{Q}$ has S real eigen values that are strictly positive.

3. The matrices $\mathbf{H}_n(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q})$ and $\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})$ are invertible for all $n \in E$.

4. $r - \mu_i - q_{ii} > 0$ for all $i \in E$ and $r > 0$.

For the calculation of the value function, the expected incoming revenue after the entry time τ plays an important role. The following lemma gives the evaluation.

Lemma 1. *The expected incoming revenue at time t is given by*

$$E\left[\int_{t}^{\infty} e^{-ru} D_{J(u)} X_u du \mid \mathcal{F}_t\right] = e^{-rt} \alpha_{J(t)} D_{J(t)} X_t,$$

where

$$\alpha_i D_i = \mathbf{e}_i^{\top} \left(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q}\right)^{-1} \mathbf{D}.$$

3 Value function

By Lemma 1, the value function at the regime i is reduced to

$$V_i(x) = \max_{\tau} E\left[e^{-r\tau}\left(\alpha_{J(\tau)} D_{J(\tau)} X_{\tau} - K_{J(\tau)}\right) \mid X_0 = x, J(0) = i\right].$$

As discussed in Jobert and Rogers (2006) and Guo and Zhang (2004), the candidate of the optimal stopping time τ must be in a form of

$$\tau = \min_{j \in E} \tau_j, \quad \tau_j = \inf\{t > 0 : X_t \ge x_j,\ J(t) = j\}.$$

We will obtain the explicit form of the value function by assuming that the order of the thresholds is

$$x_S < x_{S-1} < \cdots < x_2 < x_1 \tag{1}$$

in what follows. In case that (1) is not satisfied, the following procedure must be carried out after the regime index is interchanged appropriately. Thus, the value function is of a form of

$$V_i(x) = \begin{cases} V_i^{(0)}(x) & \text{if } x \in [x_1, \infty), \\ V_i^{(n)}(x) & \text{if } x \in [x_{n+1}, x_n), \quad (n = 1, 2, \cdots, S-1), \\ V_i^{(S)}(x) & \text{if } x \in (0, x_S). \end{cases}$$

For $x \in [x_1, \infty)$, it is optimal for the firm to start the project immediately at any regime,

$$V_i(x) = \alpha_i D_i x - K_i, \quad 1 \leq i \leq S.$$

For $x \in [x_{n+1}, x_n)$ $(n = 1, 2, \cdots, S-1)$, the firm will enter when the regime is either of $n+1, \cdots, S$, otherwise she should wait. Thus, the value function satisfies

$$\frac{1}{2}x^2\sigma_i^2\frac{d^2}{dx^2}V_i(x) + x\mu_i\frac{d}{dx}V_i(x) - rV_i(x) + \sum_{j \in E \setminus \{i\}} q_{ij}(V_j(x) - V_i(x)) = 0, \quad 1 \leq i \leq n, \tag{2}$$

and $V_i(x) = \alpha_i D_i x - K_i$ for $n+1 \leq i \leq S$. Finally, for $x \in (0, x_S)$, it obeys

$$\frac{1}{2}x^2\sigma_i^2\frac{d^2}{dx^2}V_i(x) + x\mu_i\frac{d}{dx}V_i(x) - rV_i(x) + \sum_{j \in E \setminus \{i\}} q_{ij}(V_j(x) - V_i(x)) = 0, \quad 1 \leq i \leq S.$$

We must solve simultaneous ODEs

$$\begin{aligned} \mathcal{A}_1 V_1^{(n)}(x) + \sum_{1 \leq j \leq n, j \neq 1} q_{1j} V_j^{(n)}(x) &= -\sum_{n+1 \leq j \leq S} q_{1j} V_j^{(n)}(x) \\ \mathcal{A}_2 V_2^{(n)}(x) + \sum_{1 \leq j \leq n, j \neq 2} q_{2j} V_j^{(n)}(x) &= -\sum_{n+1 \leq j \leq S} q_{2j} V_j^{(n)}(x) \\ &\vdots \\ \mathcal{A}_n V_n^{(n)}(x) + \sum_{1 \leq j \leq n, j \neq n} q_{nj} V_j^{(n)}(x) &= -\sum_{n+1 \leq j \leq S} q_{nj} V_j^{(n)}(x) \end{aligned}$$

for $x \in [x_{n+1}, x_n)$, $(n = 1, 2, \cdots, S-1)$, where

$$\mathcal{A}_i f(x) = \frac{1}{2}x^2\sigma_i^2\frac{d^2}{dx^2}f(x) + x\mu_i\frac{d}{dx}f(x) - (r - q_{ii})f(x),$$

with the value matching condition and the smooth pasting conditions at $x = x_n, x_{n+1}$. They are rewritten in a form of matrix as

$$\begin{pmatrix} \mathcal{A}_1 & q_{12} & \cdots & q_{1n} \\ q_{21} & \mathcal{A}_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & \mathcal{A}_n \end{pmatrix} \begin{pmatrix} V_1^{(n)}(x) \\ V_2^{(n)}(x) \\ \vdots \\ V_n^{(n)}(x) \end{pmatrix} = -\begin{pmatrix} q_{1,n+1} & q_{1,n+2} & \cdots & q_{1S} \\ q_{2,n+1} & q_{2,n+2} & \cdots & q_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,n+1} & q_{n,n+2} & \cdots & q_{nS} \end{pmatrix} \begin{pmatrix} V_{n+1}^{(n)}(x) \\ V_{n+2}^{(n)}(x) \\ \vdots \\ V_S^{(n)}(x) \end{pmatrix} \tag{3}$$

where $V_j^{(n)}(x) = \alpha_j D_j x - K_j, \quad j \in \{n+1, \ldots, S\}$.

For the time being, we concentrate on the equations (2) on $V_i^{(n)}(x)$ on the interval $x \in [x_{n+1}, x_n)\,(n = 1, 2, S-1)$. Since we know the solution $V_i^{(n)}(x) = \alpha_i D_i x - K_i$ for $i = n+1, \cdots, S$, the equations of the remainings $V_i^{(n)}$ for $1 \le i \le n$ are reduced to simultaneous second-order ODEs. It follows that the solution $V_i^{(n)}$ is decomposed with the general solution $\tilde{V}_i^{(n)}$ and the special solution $v_i^{(n)}(x)$ for each $i = 1, 2, \cdots, n$.

The special solution $v_i^{(n)}(x)$ is a linear function $v_i^{(n)}(x) = a_i^{(n)} x + b_i^{(n)}$. Then, the coefficients $\mathbf{a}^{(n)} = (a_1^{(n)}, \cdots, a_n^{(n)})^\top, \mathbf{b}^{(n)} = (b_1^{(n)}, \cdots, b_n^{(n)})^\top$ of the solution are given by

$$\mathbf{a}^{(n)} = \mathbf{H}_n(r\mathbf{I}_S - \mathbf{M} - \mathbf{Q})^{-1} \begin{pmatrix} \sum_{j=n+1}^{S} q_{1j}\alpha_j D_j \\ \sum_{j=n+1}^{S} q_{2j}\alpha_j D_j \\ \vdots \\ \sum_{j=n+1}^{S} q_{nj}\alpha_j D_j \end{pmatrix}, \tag{4}$$

$$\mathbf{b}^{(n)} = -\mathbf{H}_n(r\mathbf{I}_S - \mathbf{Q})^{-1} \begin{pmatrix} \sum_{j=n+1}^{S} q_{1j} K_j \\ \sum_{j=n+1}^{S} q_{2j} K_j \\ \vdots \\ \sum_{j=n+1}^{S} q_{nj} K_j \end{pmatrix},$$

where the inverse matrices are guranteed to exist by Assumption 1.

Next, we turn our eyes to the general solutions $\tilde{V}_i^{(n)}$. In order to change the variable, let us introduce auxuliary functions $\overline{V}_i^{(n)}(y) = \tilde{V}_i^{(n)}(e^y)$, $\overline{W}_i^{(n)}(y) = \frac{d}{dy}\overline{V}_i^{(n)}(y)$. Then (??) can be rewritten as a system of first-order ODEs,

$$\frac{d}{dy}\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \mathbf{\Gamma}_n \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix}, \tag{5}$$

where

$$\mathbf{\Gamma}_n = \begin{pmatrix} \mathbf{O}_n & \mathbf{I}_n \\ \mathbf{R}_n & \mathbf{C}_n \end{pmatrix}, \quad \mathbf{\Sigma}_n = \frac{1}{2}\mathrm{diag}\left[\sigma_1^2, \cdots, \sigma_n^2\right],$$

$$\mathbf{R}_n = \mathbf{\Sigma}_n^{-1}(r\mathbf{I}_n - \mathbf{H}_n(\mathbf{Q})) = \begin{pmatrix} \frac{2(r-q_{11})}{\sigma_1^2} & \frac{-2q_{12}}{\sigma_1^2} & \cdots & \frac{-2q_{1n}}{\sigma_1^2} \\ \frac{-2q_{21}}{\sigma_2^2} & \frac{2(r-q_{22})}{\sigma_2^2} & \cdots & \frac{-2q_{2n}}{\sigma_2^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2q_{n1}}{\sigma_n^2} & \frac{-2q_{n2}}{\sigma_n^2} & \cdots & \frac{2(r-q_{nn})}{\sigma_n^2} \end{pmatrix},$$

$$\mathbf{C}_n = \mathbf{I}_n - \mathbf{\Sigma}_n^{-1}\mathbf{H}_n(\mathbf{M}) = \mathrm{diag}\left[1 - \frac{2\mu_1}{\sigma_1^2}, \cdots, 1 - \frac{2\mu_n}{\sigma_n^2}\right].$$

Thus, the solution is given by

$$\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \exp\left((y - y_0)\mathbf{\Gamma}_n\right) \begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y_0) \\ \overline{\mathbf{W}}^{(n)}(y_0) \end{pmatrix}$$

with some y_0 from the boundary conditions when the exponential matrix $\exp\left((y - y_0)\mathbf{\Gamma}_n\right)$ is available. If the coefficient matrix $\mathbf{\Gamma}_n$ is diagonalizable, it is straightforward to solve the systme of ODEs (5).[1] Otherwise, one can proceed in a parallel way by making use of the Jordan normal form that is guranteed to exist for each square matrix by the theory.

The characteristic function of of $\mathbf{\Gamma}_n$, which is obtained with the knowledge of linear algebra as

$$\det \begin{pmatrix} \mathbf{O}_n - \beta \mathbf{I}_n & \mathbf{I}_n \\ \mathbf{R}_n & \mathbf{C}_n - \beta \mathbf{I}_n \end{pmatrix} = f_n(\beta) \prod_{j=1}^{n} \left(\frac{1}{2}\sigma_j^2\right)^{-1},$$

where

$$\begin{aligned} f_n(\beta) &= \det(\mathbf{\Sigma}_n \beta^2 - \mathbf{\Sigma}_n \mathbf{C}_n \beta - \mathbf{\Sigma}_n \mathbf{R}_n) = \det \begin{pmatrix} g_1(\beta) & q_{12} & \cdots & q_{1n} \\ q_{21} & g_2(\beta) & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & g_n(\beta) \end{pmatrix}, \\ g_i(\beta) &= \frac{1}{2}\sigma_i^2 \beta^2 + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)\beta - (r - q_{ii}). \end{aligned}$$

Thus, the eigenvalues are the solutions of $f_n(\beta) = 0$. In this paper we make the following assumption for simple and useful results. In case that the assumption is not satisfied, the following discussion can be accordingly modified by considering the Jordan normal form.

Assumption 2. 1. For $n = 1, 2, \cdots, S-1$, $\mathbf{\Gamma}_n$ has $2n$ distinct eigenvalues $\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}$.

2. $\mathbf{\Gamma}_S$ has $2S$ distinct eigenvalues such that $\beta_1^{(S)}, \cdots, \beta_S^{(S)}$ are strictly positive and $\beta_{S+1}^{(S)}, \cdots, \beta_{2S}^{(S)}$ are strictly negative.

By Assumption 2 there exist distinct eigenvalues $\beta_j^{(n)}$ $(1 \le j \le 2n)$. Since the upper right block of $\mathbf{\Gamma}_n$ is $\mathbf{I}_n$, the eigenvector for the eigenvalue $\beta_j^{(n)}$ must be in the form

$$\tilde{\mathbf{u}}_j^{(n)} = \begin{pmatrix} \mathbf{u}_j^{(n)} \\ \beta_j^{(n)} \mathbf{u}_j^{(n)} \end{pmatrix} \in \mathbb{R}^{2n},$$

with some non-zero vector $\mathbf{u}_j^{(n)} \in \mathbb{R}^n$ satisfying

$$\begin{pmatrix} g_1\left(\beta_j^{(n)}\right) & q_{12} & \cdots & q_{1n} \\ q_{21} & g_2\left(\beta_j^{(n)}\right) & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & g_n\left(\beta_j^{(n)}\right) \end{pmatrix} \mathbf{u}_j^{(n)} = \mathbf{0}_n. \tag{6}$$

[1]Jobert and Rogers (2006) also made use of linear algebra in the calculation of American options under regime switching.

Note that such a vector $\mathbf{u}_j^{(n)}$ exists for each j because the determinant of the coefficient matrix on the LHS of (6) is equal to $f_n\left(\beta_j^{(n)}\right) = 0$ by definition of $\beta_j^{(n)}$. Thus, $\mathbf{\Gamma}_n$ is diagonalized as

$$\begin{pmatrix} \mathbf{O}_n & \mathbf{I}_n \\ \mathbf{R}_n & \mathbf{C}_n \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \mathrm{diag}\left[\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}\right] \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix}^{-1},$$

where

$$\mathbf{U}^{(n)} = \begin{pmatrix} \mathbf{u}_1^{(n)} & \mathbf{u}_2^{(n)} & \cdots & \mathbf{u}_{2n}^{(n)} \end{pmatrix}, \quad \mathbf{B}^{(n)} = \mathrm{diag}\left[\beta_1^{(n)}, \cdots, \beta_{2n}^{(n)}\right].$$

Note that the matrix

$$\begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^{(n)} & \mathbf{u}_2^{(n)} & \cdots & \mathbf{u}_{2n}^{(n)} \\ \beta_1^{(n)}\mathbf{u}_1^{(n)} & \beta_2^{(n)}\mathbf{u}_2^{(n)} & \cdots & \beta_{2n}^{(n)}\mathbf{u}_{2n}^{(n)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{u}}_1^{(n)} & \tilde{\mathbf{u}}_2^{(n)} & \cdots & \tilde{\mathbf{u}}_{2n}^{(n)} \end{pmatrix}$$

is invertible since the eigenvalues of $\mathbf{\Gamma}_n$ are distinct so that the corresponding eigenvectors are linearly independent.

Then we can solve the system of ODEs (5) as

$$\begin{pmatrix} \overline{\mathbf{V}}^{(n)}(y) \\ \overline{\mathbf{W}}^{(n)}(y) \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)}\mathbf{B}^{(n)} \end{pmatrix} \mathrm{diag}\left[e^{\beta_1^{(n)}y}, \cdots, e^{\beta_{2n}^{(n)}y}\right] \begin{pmatrix} A_1^{(n)} \\ A_2^{(n)} \\ \vdots \\ A_{2n}^{(n)} \end{pmatrix},$$

with some constants $A_1^{(n)}, \cdots, A_{2n}^{(n)}$. By adding the special solutions, we have the vector of the value functions $\mathbf{V}^{(n)}(x) = (V_1^{(n)}(x), \cdots, V_n^{(n)}(x))^\top$ on the interval $[x_n, x_{n+1})$ for $n = 1, 2, \cdots, S-1$ given as

$$\mathbf{V}^{(n)}(x) = \mathbf{U}^{(n)}\mathbf{X}^{(n)}(x)\mathbf{A}^{(n)} + \mathbf{v}^{(n)}(x), \tag{7}$$

where

$$\begin{aligned} \mathbf{X}^{(n)}(x) &= \mathrm{diag}\left[x^{\beta_1^{(n)}}, \cdots, x^{\beta_{2n}^{(n)}}\right], \quad \mathbf{A}^{(n)} = \begin{pmatrix} A_1^{(n)} \\ A_2^{(n)} \\ \vdots \\ A_{2n}^{(n)} \end{pmatrix}, \\ \mathbf{v}^{(n)}(x) &= \mathbf{a}^{(n)}x + \mathbf{b}^{(n)}. \end{aligned}$$

Unknown boundaries $x_S < \cdots < x_1$ and unknown vectors $\mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(S)}$ will be determined by the value matching conditions, the smooth pasting conditions and the values at $x = 0$. We will investigate them by looking at x_1 first and moving downward to x_S as follows.

First, we consider the case of $n = 1, 2, \cdots, S$. The value matching conditions at $x = x_n$, $V_i^{(n)}(x_n) = V_i^{(n-1)}(x_n)$ for $i = 1, \cdots, n$ requires

$$\mathbf{U}^{(n)}\mathbf{X}^{(n)}(x_n)\mathbf{A}^{(n)} + \mathbf{v}^{(n)}(x_n) = \begin{pmatrix} \mathbf{U}^{(n-1)}\mathbf{X}^{(n-1)}(x_n)\mathbf{A}^{(n-1)} + \mathbf{v}^{(n-1)}(x_n) \\ \alpha_n D_n x_n - K_n \end{pmatrix},$$

and the smooth pasting conditions $x_n \frac{d}{dx} V_i^{(n)}(x_n) = x_n \frac{d}{dx} V_i^{(n-1)}(x_n)$ for $i = 1, \cdots, n$ requires

$$\mathbf{U}^{(n)} \mathbf{dX}^{(n)}(x_n) \mathbf{A}^{(n)} + \mathbf{a}^{(n)} x_n = \begin{pmatrix} \mathbf{U}^{(n-1)} \mathbf{dX}^{(n-1)}(x_n) \mathbf{A}^{(n-1)} + \mathbf{a}^{(n-1)} x_n \\ \alpha_n D_n x_n \end{pmatrix}$$

where

$$\mathbf{dX}^{(n)}(x) = \text{diag}\left[\beta_1^{(n)} x^{\beta_1^{(n)}}, \cdots, \beta_{2n}^{(n)} x^{\beta_{2n}^{(n)}}\right].$$

By these conditions and relationships

$$\begin{pmatrix} \mathbf{U}^{(n)} \mathbf{X}^{(n)}(x) \\ \mathbf{U}^{(n)} \mathbf{dX}^{(n)}(x) \end{pmatrix} = \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)} \mathbf{B}^{(n)} \end{pmatrix} \mathbf{X}^{(n)}(x),$$

$\mathbf{A}^{(n)}$ is represented with a function of x_n and $\mathbf{A}^{(n-1)}$ as

$$\begin{aligned} \mathbf{A}^{(n)} &= \mathbf{X}^{(n)}\left(x_n^{-1}\right) \begin{pmatrix} \mathbf{U}^{(n)} \\ \mathbf{U}^{(n)} \mathbf{B}^{(n)} \end{pmatrix}^{-1} \\ &\times \left[\begin{pmatrix} \mathbf{U}^{(n-1)} \mathbf{X}^{(n-1)}(x_n) \mathbf{A}^{(n-1)} + \mathbf{v}^{(n-1)}(x_n) \\ \alpha_n D_n x_n - K_n \\ \mathbf{U}^{(n-1)} \mathbf{dX}^{(n-1)}(x_n) \mathbf{A}^{(n-1)} + \mathbf{a}^{(n-1)} x_n \\ \alpha_n D_n x_n \end{pmatrix} - \begin{pmatrix} \mathbf{v}^{(n)}(x_n) \\ \mathbf{a}^{(n)} x_n \end{pmatrix} \right]. \end{aligned} \tag{8}$$

Similarly, for $n = 1, S$, we can obtain $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(S)}$ in a parallel way. Therefore, we can represent unknown vectors $\mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(S)}$ as functions of $x_1, \cdots, x_S$.

Furthermore, on $(0, x_S]$, we want to impose $\lim_{x \to 0} V_i^{(S)}(x) = 0$ for all i. It implies that the coefficient of $\mathbf{A}^{(S)}$ corresponding to negative eigen values $\beta_{S+1}^{(S)}, \cdots, \beta_{2S}^{(S)}$ must be zero,

$$\begin{pmatrix} \mathbf{O}_S & \mathbf{I}_S \end{pmatrix} \mathbf{A}^{(S)} = \mathbf{0}_S. \tag{9}$$

This is a set of S equations that S unknown constants $x_1, \cdots, x_S$ must satisfy. Apparently (9) is a system of complicated algebraic equations, hence they must be solved numberically. In case that the numerical solution doesn't satisfy the order condition (1), the indices of the regimes must be interchanged.

As a summary, we obtain the main result.

Theorem 1. *Suppose that $x_1, \cdots, x_S$ satisfy (1) and (9). Then the value functions are given by (7) with $\mathbf{A}^{(n)}$ given by (8).*

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